1. $(x^2+y^2+xe^y)\frac{dy}{dx} + 2xy + e^y - 1 = 0$

$\cancel{dx}(x^2+y^2+xe^y)\frac{dy}{\cancel{dx}} = (1-e^y-2xy)dx$

$(x^2+y^2+xe^y)dy + (-1+e^y+2xy)dx = 0$

$M(x,y) = -1+e^y+2xy$

$N(x,y) = x^2+y^2+xe^y$

$\frac{\partial M}{\partial y} = +\cancel{y}e^y + 2x$

A function is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x} = 2x + e^y$

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 2x + e^y$

Its solution $\int M\,dx = \int (e^y + 2xy - 1)\,dx = xe^y + 2\frac{x^2}{2}y - x + \phi(y)$

$= xe^y + x^2y - x + \phi(y)$

$\frac{\partial}{\partial y}\left[xe^y + x^2y - x + \phi(y)\right] = N$

$xe^y + x^2 + \phi'(y) = x^2+y^2+xe^y$

$\phi'(y) = y^2.$

$\phi(y) = \int \phi'(y) = y^3/3.$

Therefore solution $= xe^y + x^2y - x + y^3/3$

2. $\frac{dy}{dx} = \frac{2xy-y^2}{x^2}.$, This function is homogenous because it can be written in the form $\frac{dy}{dx} = f(y/x)$

$\frac{dy}{dx} = \frac{2xy}{x^2} - \frac{y^2}{x^2} = 2\frac{y}{x} - \left(\frac{y}{x}\right)^2.$

$\frac{dy}{dx} = 2\,y/x - (y/x)^2$ therefore homogenous

We let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}.$

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$$\frac{dy}{dx} = 2\,y/x - (y/x)^2 \qquad \Rightarrow y/x = v$$

$$v + x\frac{dv}{dx} = 2v - v^2.$$

$$x\frac{dv}{dx} = 2v - v^2 - v$$

$$x\frac{dv}{dx} = (v - v^2)\,dx$$

$$\frac{\cancel{x}\,dv}{x\,v - v^2} = \frac{(v-v^2)\,dx}{v - v^2}$$

$$\int \frac{1}{v - v^2}\,dv = \int \frac{1}{x}\,dx.$$

$$\int \frac{1}{v}\,dv - \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx.$$

$$\ln v - \frac{1}{v} = \ln x + \ln c$$

$$\ln v - 1/v = \ln x + \ln c$$

$$\ln v - (\ln x + \ln c) = 1/v.$$

$$\ln\left[\frac{v}{xc}\right] = 1/v.$$

$$\frac{v}{xc} = e^{1/v}.$$

$$v = xce^{1/v}.$$

$$y/x = Ax\,e^{x/y} = Axe^{x/y}$$

$$y = Ax^2 e^{x/y}$$

3. $x(0) = 100$

$s(t)$ = mass of salt in tank at time $t > 0$

salt concentration $2g/l$

Inflow rate $= 3\ l/min$

Outflow rate $= 3l/min$

$\frac{ds}{dt}$ = (rate of salt into tank) - (rate of salt out of tank)

$\frac{ds}{dt} = (2 \times 3)\,g/min - \frac{3s}{100}\,g/min$

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$$\frac{ds}{dt} = 6 - 0.03s$$

$$\frac{ds}{dt} = \frac{600 - 3s}{100}$$

$$100\,ds = (600 - 3s)\,dt$$

$$\frac{ds}{600 - 3s} = \frac{1}{100}\,dt.$$

$$\cancel{\int \frac{1}{600}\,ds / - \int \frac{1}{3s}\,ds = \int \frac{1}{100}\,dt}$$

$$\cancel{\frac{s}{600}} \; \cancel{+} \quad -\frac{1}{3}\left[\ln|600 - 3s|\right] = \frac{t}{100} + c$$

$$\ln|600 - 3s) = \frac{-3t}{100} + c$$

$$600 - 3s = e^{-3t/100} \cdot e^{c}$$

$$600 - 3s = A e^{-3t/100}$$

$$\frac{-\cancel{3}s}{\cancel{-3}} = \frac{A e^{-3t/100} - 600}{-3}$$

$$S(t) = \frac{600 - Ae^{-3t/100}}{3}$$

For pure water $\quad s(0) = 0$

$$0 = \frac{600 - A}{3} \Rightarrow A = 600$$

$$s(t) = 200 - 200\,e^{-3t/100}$$

(b) $s(t)$ is a decreasing function because it is a negative exponent of time.

c) as $s(t)$ tends to 200 the constant because as $t \to \infty$ the exponent part tends to zero.

d) $s(t) = 200 - 200\,e^{-3t/100} = 200 - 200\,e^{-0.03t} = 200 - 200(e^{+0.03} \times e^{-t})$

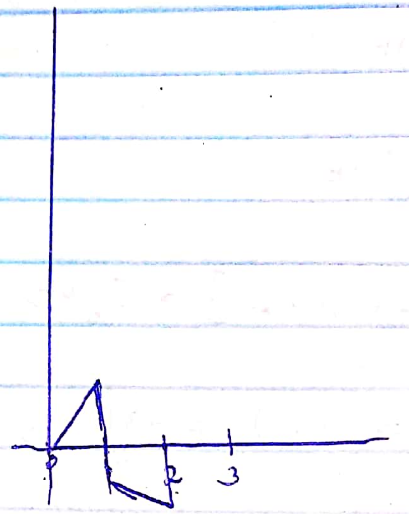
$$= 200 - 200\,(1.03\,e^{-t})$$

$$s(t) = 200 - 206(e^{-t})$$

4 a)

$$y(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$2-t = -t+2$



In terms of step functions

$y(t) = t\,[u_0(t) - u_1(t)] + 2-t\,[u_1(t) - u_2(t)]$

from time 0 - 1s the function t is turned as explained or shown above and when t = 1 to t = 2s, function 2-t is turned on.

$y(t) = t\,u(t-0) + (2-t)[u(t-1) - u(t-2)]$

$y(t) = t(u(t) - u(t-1)) + (2-t)[u(t-1) - u(t-2)]$

5. b) $(t + e^{2t})^2$.

$\mathcal{L}(t+e^{2t})^2$

but $(t+e^{2t})^2 = (t+e^{2t})(t+e^{2t})$

$= t^2 + 2te^{2t} + e^{4t}$

$\mathcal{L}^{-1}[t^2 + 2te^{2t} + e^{4t}] = \mathcal{L}t^2 + \mathcal{L}\,2te^{2t} + \mathcal{L}e^{4t}$

$= \frac{2}{s^3} + 2.\frac{1}{(s-2)^2} + \frac{1}{s-4}$

$\mathcal{L}t^n = \frac{n!}{s^{n+1}} + \mathcal{L}t^n e^{at} = \frac{n!}{(s-a)^{n+1}} + \mathcal{L}e^{at} = \frac{1}{s-4}$ $= \frac{2}{s^3} + \frac{2}{(s-2)^2} + \frac{1}{s-4}$

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5. (a) $\mathcal{L}^{-1}\ \dfrac{2s+4}{s^2-4s+12.}$

$$\frac{2s+4}{s^2-4s+12}$$

Roots of denominator

$$r_1, r_2 = \frac{4 \pm \sqrt{4^2-12\times 4}}{2}$$

$$= 2 \pm \frac{i6}{2}$$

$$= 2 \pm 3i$$

Simplify the denom by completing the square

$$s^2-4s+12$$

$$s^2-\left(\tfrac{4}{2}\right)^2 s+12-\left(\tfrac{4}{2}\right)^2$$

$$s^2-4s+8+4.$$

$$(s^2-4s+4)+8$$

$$(s-2)^2+8$$

$$\frac{2(s+2)}{(s-2)^2+8} = 2\left[\frac{s+2}{(s-2)^2+8}\right. = \frac{2(s+2)}{(s-2)^2+\sqrt{8}}$$

$$\mathcal{L}^{-1}\left[\frac{2(s+2)}{(s-2)^2+\sqrt{8}}\right] = \sqrt{2}\,e^{2t}/(2\sin 2\sqrt{2}t) + \sqrt{2}\cos(2\sqrt{2}t)$$

(4). (b) $\dfrac{s}{(s-4)(s^2+1)}$

According to convolution theorem

$$(f*g)t = \int_0^t f(t-v)\,g(v)\,dv.$$

$\dfrac{s}{(s-4)(s^2+1)}$ is a product of 2 Two laplace transform

$$\frac{s}{s-4}\cdot\frac{1}{s^2+1}$$ ~~in sin at = sin t~~

$$\mathcal{L}^{-1}\left[\frac{s}{s-4}\cdot\frac{1}{s^2+1}\right] = \mathcal{L}^{-1}\left[\frac{s}{s-4}\right]\cdot\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \delta t + 4e^{4t}\cdot\sin t$$

For $t > 0$ $\delta t = 0$

therefore $4e^{4t}\sin t$

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6. $E = 100 \cos 10t$

$L = 1\ H$

$R = 40\ \Omega$

$C = 16 \times 10^{-4}\ F$

$Q(0) = 0$

$i(0) = 0$

This is a series connection

$$\alpha = \frac{R}{2L} = \frac{40}{2 \times 1} = 20$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{16 \times 10^{-4}}} = 25$$

$\alpha < \omega_0$, therefore the system is underdamped

Generic soln

$$i(t) = e^{-\alpha t}\left[B_1 \cos \omega_d t + B_2 \sin \omega_d t\right]$$

$$= e^{-20t}\left[B_1 \cos 15t + B_2 \sin 15t\right]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{25^2 - 20^2} = 15$$

$i(0) = 0 = B_1$

$\therefore\ i(t) = e^{-20t} B_2 \sin 15t$ — current at any instant t

$di/dt = e^{-20t} \cdot 15 B_2 \cos 15t \qquad i'(0) = 0 = e^{-20t}$

We know $Q = CV \qquad V = 100 \cos(10t)$ Finally because of the parallel connection

$Q =$

$Q = 16 \times 10^{-4}\ F \cdot 100 \cos 10t.$

$Q(t) = 0.16 \cos 10t$

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